4. Reduced-order Models

The exact description of systems with distributed elements is often quite complex and too difficult to solve. For this reason, the approach using the approximation by reduced elements is frequently used. Such a reduced model is composed by reduced elements that replace the distributed elements in such a way that have, at the point to which they are reduced, the same effect as the original distributed system. The reduced system, having only one reduced mass m_r and one reduced compliance c_r, has only one resonant frequency that corresponds to the first mode of vibration of the given structure. This approximation is thus useful only for a limited frequency range starting at low frequencies and ending at the vicinity of the first mode of vibrations.

We will examine in this chapter basic structural elements with the goal to connect their detailed mechanical behaviour to equivalent lumped-circuit representations. These structures are axially loaded beams, transversely loaded beams, and plates. We will also consider the influence of specific effects on the behaviour of these structures.

The general rule that can be adopted for reduced elements calculation is based on the knowledge of the deformation curve of a structure. If we are looking for a reduced mass, we compare the kinetic energy of the distributed mass of the analyzed structure with that of the corresponding reduced mass having the same displacement as the point of the structure to which the reduction is done. The point to which the reduction is done is usually the point of maximal deflection. Equality relation for kinetic energies is:

$$\frac{1}{2} m_r \left|v_0\right|^2 = \int_m \frac{1}{2} \left|v\right|^2 dm \tag{1}$$

where *v* is the speed of a structure mass element, *dm* and v_0 is the speed at the point of the reduction. If we suppose the harmonic movement, with the displacements *u* and u_0, corresponding to the speeds *v* and v_0, the reduced mass will be:

$$m_r = \int_m \left(\frac{u(x,y)}{u_0} \right)^2 dm \tag{2}$$

The reduced compliance can be obtained from the static deformation by the load *F* in the point of reduction. If this force gives rise to the deflection u_0, the reduced compliance is given by:

$$c_r = \frac{u_0}{F} \tag{3}$$

An alternative approach can be used in the case when natural frequencies of the examined structure are estimated. Supposing that we know one of two reduced elements, the second can be calculated from the simple relation:

$$f_r = \frac{1}{2\pi} \sqrt{\frac{1}{m_r c_r}} \tag{4}$$

The fact that the first approach is based on the static solution of system behaviour whereas the second approach is based on the dynamical behaviour can explain possible differences.

4.1 Axially Loaded Beams

The simplest structure is a slender beam with rectangular cross section of the length *L*, the width *w*, and the height *h*. We show the beam with one fixed end in the Figure 4.1. In this typical one-dimensional problem, the stress corresponding to the axial load force $T = F/wh$ creates the strain $S = DL/L$.

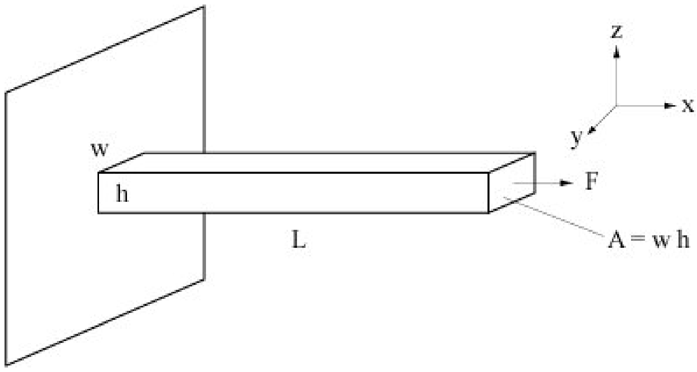

Figure 4.1 Axially loaded beam.

Based on the basic definition of elasticity, we can obtain the relation for the strain as:

$$S = \frac{T}{E} = \frac{F}{whE} \tag{5}$$

In the static case, the force, *F* and the deformation *DL* are constant along the beam. The static compliance, c_s, and the spring constant, *k*, of the beam functioning as a spring element in a mechanical structure is then defined as:

$$c_s = \frac{1}{k} = \frac{\Delta L}{F} = \frac{L}{whE} \tag{6}$$

The behaviour of the beam in the dynamic vibrations must be analyzed, based on the solution of the equation of motion corresponding to this case. The equation of motion for the longitudinal displacement u_x has the form:

$$\frac{\partial^2 u_x}{\partial x^2} = \frac{1}{c_E^2}\frac{\partial^2 u_x}{\partial t^2} \tag{7}$$

The solution for displacement u_x corresponding to the boundary conditions shown in Figure 4.1 is:

$$u_x = u_L \frac{\sin kx}{\sin kL} \tag{8}$$

In the Expression (8), u_L is the displacement at the free end of the beam, *k* is the wave number, and *x* is the coordinate. The strain along the beam is defined by the relation:

$$S = \frac{\partial u_x}{\partial x} \tag{9}$$

We can obtain from the Hooke's law the stress corresponding to the strain as:

$$T = ES = Ek\frac{\cos kx}{\sin kL}u_L \tag{10}$$

Further, we can obtain by considering $x=L$, the stress T_L and hence the force F_L, at the free end of the beam, and finally, we can get for the mechanical impedance, Z_{mL} at this point, if supposing the harmonic movement, the following expression:

$$Z_{mL} = \frac{F_L}{v_L} = \frac{T_L A}{v_L} = EAk \frac{1}{j\omega \, tg \, kL} \tag{11}$$

The Expression (11) can be written by replacing the goniometric functions by its decomposition to the series as:

$$\frac{1}{Z_{mL}} = \sum_{n=1}^{\infty} \frac{1}{j\omega m_e + \dfrac{1}{j\omega c_n}} \tag{12}$$

where m_e is the equivalent mass and c_n is the compliance equivalent to the mode of vibrations n as given in following expressions:

$$m_e = \frac{\rho AL}{2}, \quad c_n = \frac{8c_s}{\pi^2 (2n-1)^2}, \quad \omega_n = \frac{\pi(2n-1)c_E}{2L} \tag{13}$$

The mechanical impedance of an axially loaded beam can be thus represented for each mode by the mass, the compliance, and optionally the resistance connected in series. The complete description of the beam is the parallel assembly of branches corresponding to each mode as shown in Figure 4.2.

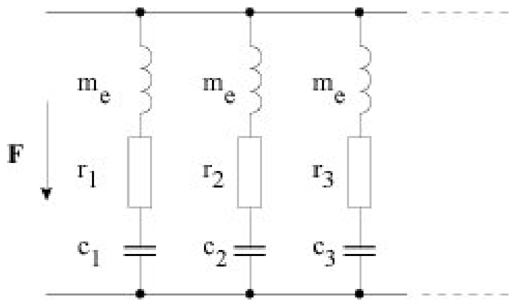

Figure 4.2 Equivalent circuit for longitudinal wave in a beam.

4.2 Transversely Loaded Beams

We will assume, in this paragraph the beams of rectangular cross-section with the length much greater than either of their transverse dimensions. Before we will consider bending-type deformations resulting from the transverse loading of beams we will examine types of supports and types of loads on structures that can bend. The basic kinds off supports that are mathematically defined in the Chapter 2 are shown in Figure 4.3. The functional equivalent of all these types of support can be encountered in MEMS devices.

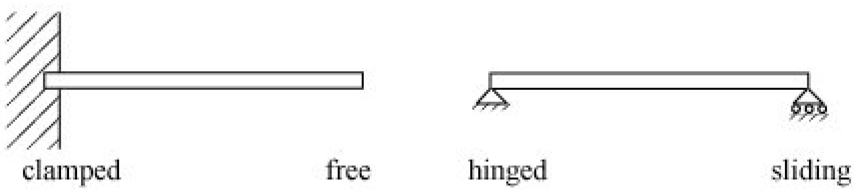

Figure 4.3 Different types of structure support.

The clamped (also called fixed or built-in) support doesn't permit the horizontal and vertical movement at the support nor a non-zero slope at the support. There is no constraint on the displacement or the slope in the case of free end. A beam with a free end is called a cantilever beam. The hinged (also called pinned or simple support) fixes both the vertical and horizontal position of the beam end, but does not restrict its slope. The sliding support fixes the vertical position of the beam, but does not restrict its horizontal position and slope.

Figure 4.4 shows two different types of external transverse load that can be applied to beams.

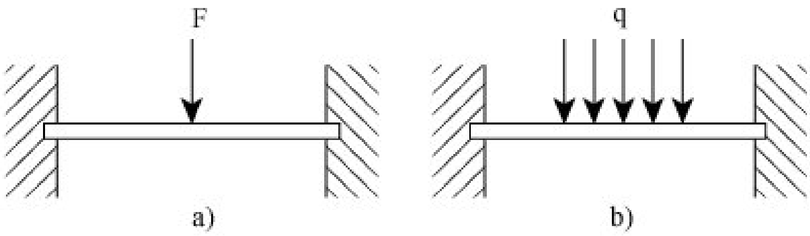

Figure 4.4 Different types of loads: a) Point load, b) Distributed load.

A point load *F* can be applied at any position along the beam length. We will suppose that *F* is the total force acting at a position x along the length of a beam, uniformly distributed across its width. Sometimes, if the force per unit width, *F'* is specified, the total force *F = F'w*. A distributed load *q* can be applied to a portion of the beam length. The symbol *q* is used for the force per unit length of beam, uniformly distributed across its width. If a portion of a beam is loaded by a uniform pressure *P*, the equivalent beam load can be expressed as *q = Pw*.

We can apply the general rules described at the beginning of this chapter for the calculation of a beam reduced parameters. We need to know the deformation curve corresponding to each specific case, given by a type of support and a type of load, for the reduced mass calculation. The compliance of a beam is calculated from the deflection caused by a known force.

For a **cantilever,** we can obtain the displacement along its length as:

$$u(x) = \frac{F(L-x)^2}{6EI}(2L+x) \tag{14}$$

The displacement of the free end (*x=0*: point at which reduced elements are calculated) is equal to:

$$u_0 = \frac{FL^3}{3EI} \tag{15}$$

The reduced mass of a cantilever can be obtained from Equation (2) after the substitution from Equations (14) and (15) as:

$$m_r = 0.236\, m_L L = 0.236m \tag{16}$$

The term m_L is mass per unit length, *m* is beam total mass. The reduced compliance will be obtained by comparing the Equations (3) and (15), which gives the following value:

$$c_r = \frac{L^3}{3EI} \tag{17}$$

For a cantilever with a rectangular cross-section this gives:

$$c_r = 4\frac{L^3}{Ewh^3} \tag{18}$$

The displacement along the length of a **beam simply supported** on his both ends is:

$$u(x) = \frac{Fx}{48EI}(3L^2 - 4x^2) \tag{19}$$

We can obtain, similarly as in the preceding case, the reduced elements corresponding to this case. For a beam of a rectangular cross-section we thus have:

$$m_r = 0.485\,m\,, \quad c_r = \frac{L^3}{4Ewh^3} \tag{20}$$

Finally, the reduced elements of a **beam clamped** on his both ends, having a rectangular cross-section, are given by following expressions:

$$m_r = 0.371\,m\,, \quad c_r = \frac{L^3}{16Ewh^3} \tag{21}$$

Similar results can be obtained if we calculate the frequency of the first mode of vibration from the general expression [PLU 99]:

$$f_i = A_i\sqrt{\frac{EI}{m_L L^4}} \tag{22}$$

The values of coefficients A_i for different types of support are given in Table 4.1.

Type of Beam Support	**Modes**		
	1	**2**	**3**
Clamped-free	0.56	3.51	9.82
Clamped-clamped	3.56	9.82	19.26
Hinged-hinged	1.57	6.29	14.15
Clamped- hinged	2.45	7.96	16.55

Table 4.1 Coefficients A_i for calculation of modes of vibrations of beams.

If a **beam containing several layers of different materials** is analyzed, the bending stiffness *EI* and mass *m* terms must be replaced with composite bending stiffness $\overline{EI}$ and composite mass $\overline{m}$. These terms are defined as [SME 00]:

$$\overline{EI} = \sum_{i=1}^{N} E_i I_i\,, \quad \overline{m} = \sum_{i=1}^{N} m_i \tag{23}$$

where N is the number of layers in the composite beam, and E_i, I_i, m_i are properties of the i^{th} layer in the cross-section of the beam.

The **residual stress** has an important effect to the mechanical behaviour of MEMS structures. This stress appearing after the fabrication process can be decomposed into two components. The **thermal mismatch stress** is created by the deposition of thin layers to the substrate at the higher temperature and by cooling to room temperature. The **intrinsic stress** covers all other types of stresses as that occurring during chemical reactions, doping, epitaxial growth, evaporation, sputtering, and other processes. Residual stresses can have three important effects on beams. Residual stresses in some cases modify the bending stiffness; nonuniform residual stresses can cause beams to curl; and compressive residual stresses are source of beam buckling. Residual stresses and thus their effect on the beam properties can be modified by thermal processes such as annealing.

4.3 Transversely Loaded Plates

The bending behaviour of plates can be understood with a direct extension of the bending of beams. Unlikely to beams, the transverse contraction is equal to zero in directions parallel to a plate surface during its bending or axial elongation. The stress-strain relation in plates for a longitudinal movement is written as:

$$T_x = \frac{E}{1-\sigma^2} S_x \tag{24}$$

where *s* is the Poisson ratio. The quantity $E/(1\text{-}\sigma^2)$ is called plate modulus.

We can use the same approach for the approximation of a plate by its reduced elements. From the deformation curve corresponding to a given plate support and a type of load, we can calculate the reduced mass, and from the displacement caused by this load we can obtain the reduced compliance. As the analysis of plate deformations is more difficult than in the case of a bar, we will do the reduction only for two symmetrical cases of a circular plate. We will suppose for this reason that a small-displacement deformation curve of a **simply supported circular plate** can be approximated by sphere and thus can be described by a second order expression of a following form:

$$u(r) = u_0 \left[1 - \left(\frac{r}{R} \right)^2 \right] \tag{25}$$

The symbol *R* is a plate radius. The reduced mass, m_r for this case is obtained from the Expression (2) after substituting the displacement given by the Expression (25). The reduced compliance, c_r is obtained from the displacement caused by the point force at the centre of a plate that can be for approximated by the following value:

$$u_0 = \frac{3FR^2}{4\pi h^3} \frac{\sigma^2 (1-\sigma)(3+\sigma)}{E} \tag{26}$$

Both reduced elements are given by following expressions:

$$m_r = 0.333\, m\,, \quad c_r = 0.55 \frac{R^2}{Eh^3} \tag{27}$$

For a **clamped circular plate** we will suppose as a first approximation a deformation curve of a form:

$$u(r) = u_0 \left[1 - 2\left(\frac{r}{R}\right)^2 + \left(\frac{r}{R}\right)^4 \right] \tag{28}$$

The displacement at the centre corresponding to a force acting at the same point is:

$$u_0 = \frac{3FR^2}{4\pi h^3}\frac{1-\sigma^2}{E} \tag{29}$$

The reduced elements corresponding to this case are given by following expressions:

$$m_r = 0.2\ m\ , \quad c_r = 0.217\frac{R^2}{Eh^3} \tag{30}$$

If a thin plate of constant thickness vibrating transversely has a simple geometric form, the frequency and forms of vibrations modes can be determined analytically. The frequency of the i^{th} mode of vibration can be calculated from the expression of the general form:

$$f_i = C_i\sqrt{\frac{B}{\rho h a^4}} \tag{31}$$

The term B, called flexural rigidity of the plate is defined by Expression 2.24. The symbol *a* is used for diameter in the case of circular plate and for the side in the case of square plate. The values of coefficients C_1 for different forms of thin plates with different types of support are given in Table 4.2.

Form of Plate	**Type of Support**	**C_1**
Circular	Clamped on Edge	6.23
	Hinged on Edge	3.10
	Clamped in Centre	2.29
Square	Clamped on 1 Edge, Free on 3 Edges	0.53
	Clamped on 4 Edges	5.47
	Hinged on 4 Edges	3.00

Table 4.2 Coefficients C_i for calculation of modes of vibrations of plates.

The approximations we have used above were based on a small-displacement assumption and are valid only in some extent. If plate deflection is comparable or bigger than a plate thickness, more general expression must be used. In some applications, the relation between the plate displacement and the pressure difference *P*, instead of the point force, is of interest. The expression, which relates the deflection in the centre to the applied pressure, is the following:

$$\frac{PR^4}{Eh^4} = \frac{16}{3(1-\sigma^2)}\frac{u_0}{h} + \frac{7-\sigma}{3(1-\sigma)}\frac{u_0^3}{h^3} \tag{32}$$

This equation is nonlinear in u_0 and cannot be solved for u_0. The first term represents the stiffness associated with the bending of the thin plate. The second term represents the stiffness associated with the stretching of the plate that introduces nonlinearity. For small-deflection cases, if $u_0 < h$, the second term can be neglected and simplified expression is:

$$u_0 = \frac{3PR^4}{16h^3}\frac{1-\sigma^2}{E} \tag{33}$$

4.4 Related Reading

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