2. Distributed Elements Systems

In the transducer design, especially in the high frequency range, we meet problems where the dimensions of the transducer dynamic structure are comparable to the wavelength. We must thus consider wave propagation through such a system for its precise description. The behaviour of distributed elements systems is determined by the dimensions of a system, the physical constants and the system boundary conditions. The description of these systems leads to solutions expressed by series of modes or natural frequencies. The knowledge of these frequencies is important especially in large-band systems. If we are interested only in a narrow frequency band in the vicinity of one of the modes we can often describe a system by a simplified reduced-order model.

2.1 One-Dimensional Systems

A one-dimensional system with distributed parameters is a system whose dimensions on the plane perpendicular to the longitudinal axis are small compared with the wavelength. The cross-section of the system and other parameters, such as coefficient of absorption, or electric resistance per length unit, can be functions of the length coordinate. Systems with identical properties along their whole length are called uniform. A cable line used in telecommunications is a typical one-dimensional electric system. A long narrow tube is a case of one-dimensional acoustic system. In mechanics, one-dimensional systems are represented by strings and bars. Strings are assumed to be perfectly flexible; their stiffness is obtained by means of tension by external forces parallel to the string length. Bars possess their own internal stiffness. In this chapter, we will describe some cases of one-dimensional solid and gas-filled structures that can be encountered in the transducer design.

Longitudinal Waves in a Bar

One of the simplest one-dimensional structures is a slender bar with a uniform cross-section loaded with a uniform stress on its end faces and with the dimensions shown in Figure 2.1.

Figure 2.1 A long slender bar with an axial load.

The material properties of the bar are the Young's modulus *E* and the density *r*. If we consider the loading force F equal to zero, we can write the equation of motion for the longitudinal displacement u_x in the form:

$$\frac{\partial^2 u_x}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 u_x}{\partial t^2} = \frac{1}{c_L^2}\frac{\partial^2 u_x}{\partial t^2} \tag{1}$$

We can deduce from the Equation (1) the value of the longitudinal or extensional or compressional velocity:

$$c_E = \sqrt{\frac{E}{\rho}} \tag{2}$$

We will see later that this velocity is different from that in thin plate or in bulk material. The Equation (1) was written under the assumption that there are no lateral displacements. Generally, this assumption is not valid but can be approximated in the case of very thin bar. Several authors searched the approximation for the extensional velocity in bars of lateral dimensions comparable to the wavelength. We will show here the expression approximating the extensional velocity by considering the lateral displacements:

$$c_D = \sqrt{\frac{E}{\rho}} \left(1 - \frac{1}{2} k_D^2 \sigma^2 I_M\right) \qquad (3)$$

In the expression (3), $k_D = 2\pi f / c_D$ is the wave number, $\sigma = u_y / u_x$ is the Poisson ratio and I_M is the moment of inertia of the cross-section related to the axis of the bar. For rectangular cross-section corresponding to the Figure 1.1, we have:

$$I_M = \frac{w^2 + h^2}{12} \qquad (4)$$

A mechanical plane wave in a slender bar can be modelled by using the analogy with the electromagnetic wave propagating along long lines. We will suppose that a bar is composed of short segments of an infinitesimal length *dx*. Each of these segments has a mass *dm* and a compliance *dc*:

$$dm = \rho A dx \quad , \qquad dc = \frac{dx}{EA} \qquad (5)$$

The corresponding model is shown in Figure 2.2.

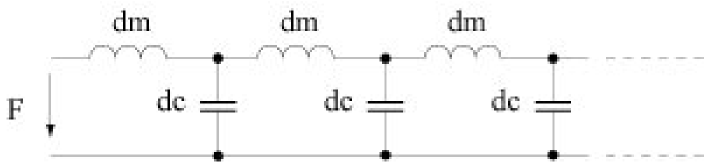


Figure 2.2 Equivalent circuit for plane waves in a thin bar.

The input mechanical impedance of a slender bar can be, with the aid of this model, expressed as:

$$Z_m = j\omega dm + \cfrac{1}{j\omega dc + \cfrac{1}{j\omega dm + \cfrac{1}{j\omega dc + \cdots}}} \qquad (6)$$

It is sometimes more convenient to work with a model based on the solution of a plane wave propagation in a slender bar. This model, unlikely to that shown in Figure 2.2, has a finite number of elements but the value of each element is frequency dependent. The general form of a model corresponding to the simple lossless structure in Figure 2.1 is shown in Figure 2.3.

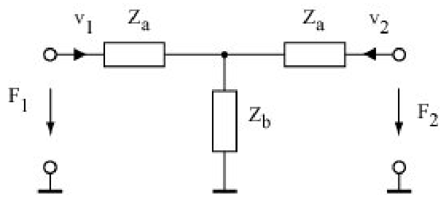


Figure 2.3 Plane waves in a thin bar modelled by the equivalent circuit with non-constant elements.

The impedances Z_a and Z_b in this circuit are:

$$Z_a = jc\rho A \tan\frac{\omega L}{2c_D} \quad , \quad Z_b = c\rho A \frac{1}{j\sin(\omega L/c_D)} \tag{7}$$

Transverse Waves in a Bar

There are a lot of different configurations in MEMS based devices using transversal movement of bars. The mode of transversal vibrations depends on the type of bar excitation. Shear, torsion and bending are the basic cases of transversal vibrations of a bar.

In a **shear motion**, the particles vibrate perpendicularly to the direction of propagation. This type of waves can be generated in a bar if the dimensions satisfy the empirical condition:

$$w, h < L/2 \tag{8}$$

The equation of motion in the case of shear waves has the form of a longitudinal wave equation where the longitudinal displacement is replaced by the displacement perpendicular to the bar axis. The velocity of shear waves is given as:

$$c_S = \sqrt{\frac{G}{\rho}} \tag{9}$$

where G is the shear elastic modulus that can be expressed with the aid of the Young's modulus E and the Poisson ratio s as:

$$G = \sqrt{\frac{E}{2(\sigma+1)}} \tag{10}$$

A torsional motion involves the propagation of a twisting disturbance along the length of the bar. For a bar of constant cross-section, the equation of motion has the same form as in the longitudinal case. The only difference is that the longitudinal displacement is replaced by the angle at which the bar is twisted relative to its equilibrium state. The velocity of torsional waves is given as:

$$c_T = \sqrt{\frac{C}{\rho J}} \tag{11}$$

where J is the polar moment of inertia, and C is a constant which depends on the geometry of the cross-section. For a circular cross-section there are simple torsional waves with speed equal to a shear wave c_S, independent of the radius R.

Bending waves are more complex than the preceding modes. This is due to the fact that in this mode, both longitudinal and transverse components are coupled. The situation on a differential angular segment of a beam that has become bent in response to the application of a transverse load is shown in Figure 2.4.

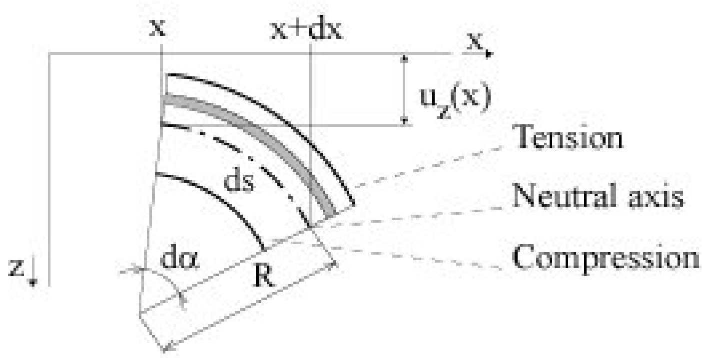


Figure 2.4 A segment of a beam in pure bending (not in scale).

The lower part of the segment is compressed and the upper part is stretched. Between the top and the bottom of the bar there is a neutral axis whose length remains unchanged. The neutral axis coincides with the central axis of the bar if the cross-section is symmetrical about a horizontal plane. It can be shown that a bent beam has a linearly distributed internal stress in the *x* direction. Its value depends on the *z* coordinate:

$$T_x = -\frac{zE}{R} \tag{12}$$

The total **internal bending moment** M can be obtained by calculating the first moment of the distributed internal stress. For a rectangular cross-section this gives:

$$M = \int_{-h/2}^{h/2} w\, z\, T_x dz = -\frac{1}{12} wh^3 \frac{E}{R} = -\frac{EI}{R} \tag{13}$$

The quantity denoted by the symbol *I* is the **moment of inertia** of the beam cross-section taken with respect to the axis parallel to the width, lying in the neutral plane. The product *EI* is called as **bending stiffness**. For a rectangular cross-section the moment of inertia is given by:

$$I = \int_{-h/2}^{h/2} z^2 w dz = \frac{1}{12} wh^3 \tag{14}$$

It is convenient to introduce the notion of **radius of gyration** of an area *g*:

$$g^2 = \frac{1}{A}\int z^2\, dA = \frac{I}{A} \tag{15}$$

The **radius of curvature** R is not in general constant but is rather a function of the position along the neutral axis. Generally, for MEMS devices, displacements u_z are limited to small values and $\partial u_z/\partial x << 1$. Then we can use the approximate relation for the estimation of the radius R:

$$R = \frac{\left[1 + (\partial u_z/\partial x)^2\right]^{3/2}}{\partial^2 u_z/\partial x^2} \approx \frac{1}{\partial^2 u_z/\partial x^2} \tag{16}$$

Many efforts had been devoted to obtaining and understanding the solution of the transversally vibrating beam problem. There exist four principal beam theories that describe beam behaviour with different degree of precision. The most detailed, Timoshenko model, takes into account *bending moment, lateral displacement, shear deformation* and *rotary inertia*. The equation of motion in its simplest form can be written as:

$$\frac{\partial^4 u_x}{\partial x^4} = -\frac{1}{g^2 c_E^2}\frac{\partial^2 u_x}{\partial t^2} \tag{17}$$

where c_E is the extensional velocity and A the beam cross-section area. One significant difference between this differential equation and the simpler equation for longitudinal waves in the rod is the presence of a fourth partial derivative, rather than a second partial, with respect to x. As a result of this difference, transverse waves do not travel along the bar with a constant speed and unchanging shape. Velocity of the propagation c_B of a flexural wave of sinusoidal vibrations with the angular frequency *w* can be written in the form:

$$c_B^2 = c_E^2 \frac{(k_B g)^2}{1+(k_B g)^2} \tag{18}$$

where $k_B = \omega/c_B$. Figure 2.5 shows as an example a velocity curve corresponding to a steel bar.

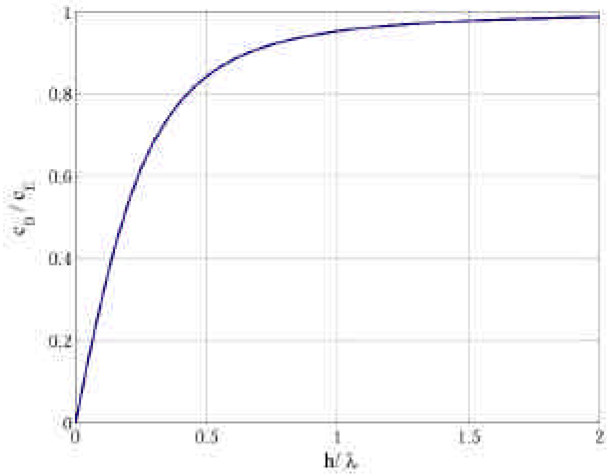


Figure 2.5 Phase velocity of flexural waves in a bar.

In many MEMS applications, we can suppose that the wavelength is large compared to transverse dimensions of the bar, that is $k_B g << 1$. In that case we obtain approximately:

$$c_B = \sqrt{c_L \omega g} \tag{19}$$

Unlike the speed of longitudinal or transversal wave, that can be considered as independent of frequency, the bending wave propagates with the speed that is proportional to $(\omega)^{1/2}$.

Thermal conductivity in the transverse cross-section may be of interest in some MEMS applications. A neutral line divides this cross-section into a region of compression and a region of extension. There then arise two neighbouring regions with raised and lowered temperatures, between which a heat exchange takes place. It can be shown that there is a frequency f_T at which a maximum of thermal absorption can be observed. For a rectangular bar of thickness *h* this frequency is:

$$f_T = c_E^2 \frac{\pi\gamma}{2hC\rho} \tag{20}$$

where γ is the coefficient of heat conductivity, C is the specific heat, and ρ is the density.

The general solution of the equation of motion can be obtained by employing hyperbolic and trigonometric identities as:

$$u_x = \cos(\omega t + \phi)\left[A\cosh\frac{\omega x}{c_B} + B\sinh\frac{\omega x}{c_B} + C\cos\frac{\omega x}{c_B} + D\sin\frac{\omega x}{c_B}\right] \tag{21}$$

where A, B, C, and D are real constants evaluated through the application of initial and boundary conditions, specific to the nature of the beam support are defined. Since a general solution contains four arbitrary constants, two pairs of boundary conditions must be specified at the ends of the bar. The boundary conditions may be defined either for the displacement u, for the slope $\frac{\partial u}{dx}$, for the inertia $\frac{\partial^2 u}{dx^2}$, or for the shear $\frac{\partial^3 u}{dx^3}$. If, for example, the end of the bar is rigidly clamped, both the displacement and the slope at the end must be zero at any time t. Similarly, other forms of boundary conditions are shown in Table 2.1.

Bar End	u	$\frac{\partial u}{\partial x}$	$\frac{\partial^2 u}{\partial x^2}$	$\frac{\partial^3 u}{\partial x^3}$
Clamped	0	0		
Free			0	0
Hinged	0		0	
Sliding		0		0

Table 2.1 Boundary conditions for flexural waves in a bar.

By applying the boundary conditions to the general solution given by the expression (21), we can obtain the frequencies corresponding to the allowed modes of vibration for a specific beam support. The table 2.2 gives the transcendental equations that must be solved in order to obtain the modes of vibrations. From these equations we find the wave number K_n and hence the mode frequencies f_n of the system. The modes are given by the expression:

$$f_n = \frac{\pi g}{2L^2}\left(\frac{E}{\rho}\right)^{1/2} q_n^2 \tag{22}$$

The quotients q_n=2n/c_B=2L/λ_n depend merely on the manner of fixing the bar and on the order of the harmonic. The values of these quotients for the first few modes and for different beam supports are given in Table 2.2.

Type of Beam Support	**Mode Equation**	**Quotients q_n**
Free-free Clamped-clamped	coshKL cosKL = 1	3.0112, 5, 7, 9, …
Clamped-free	coshKL cosKL = -1	1.194, 2.988, 5, 7, …
Free-hinged Clamped-hinged	tanhKL – tanKL = 0	
Free-sliding Clamped-sliding	tanhKL + tanKL = 0	
Hinged-hinged	sinKL = 0	2, 2.828, 3,464, 4, …

Table 2.2 Different types of support of flexural vibrating bar.

It is generally worthwhile to make some simplifying assumptions regarding the dimensions and boundary conditions. A commonly made assumption is that, although the edge of the beam is clamped (built-in) to the support, it is approximated as being simply supported (hinged). It is a reasonable approximation if the thickness of the beam, h, is much smaller than its length, L. This condition allows rotational and longitudinal displacement at the edge.

One-dimensional Acoustic Systems

In MEMS devices, solid mobile structure elements interact with gas-filled elements, such as tubes, slots, cavities, etc. Their influence on the system behaviour is very important, especially in term of damping and stiffness. Generally these acoustic elements can be considered as lumped if we take into account typical dimensions of MEMS structures comparing to wavelengths of low frequency signals. In some other cases of either larger structure dimensions or higher frequency range, the wave propagation through the acoustic element must be considered.

We will show here the possible approach to the modelling of a plane wave in the cylindrical tube. The model of a tube with an acoustic plane wave is similar to that of a bar with a mechanical plane wave shown in Figure 2.2. One differential segment of this model respecting viscous losses dr_{av} and thermal losses dr_{at} is shown in Figure 2.6.

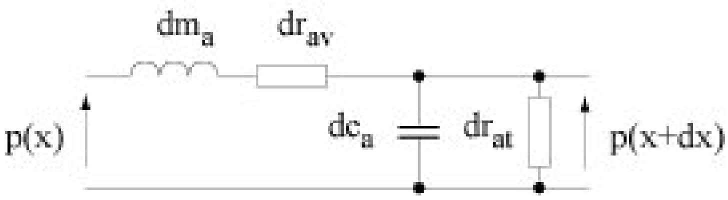


Figure 2.6 Equivalent circuit of a segment of a tube with acoustic plane wave.

The elements of this equivalent circuit valid for a capillary tube of radius *a* have the following values:

$$dm_a = \frac{4\rho_0}{3\pi a^2}dx \quad , \quad dr_{av} = \frac{8\mu}{\pi a^4}dx$$
$$dc_a = \frac{\gamma\pi a^2}{\rho_0 c_0^2}dx \quad , \quad \frac{1}{r_{at}} = (\gamma - 1)\frac{\omega^2 \pi a^4 c_P}{8\lambda c_0^2}dx \tag{23}$$

where c_0 is the sound speed in gas, l is the coefficient of thermal conductivity of gas, c_p is the thermal capacity at constant pressure, r_0 is the gas density, μ is the coefficient of viscosity and g is the ratio of specific heats at the constant pressure and at a constant volume.

2.2 Two-Dimensional Systems

Systems possessing two dimensions that are large in comparison to the third one are represented by plates and membranes. The theory of vibrations of plates and membranes constitutes a large branch of theoretical and applied mechanics, and there is no need to enter into a detailed description here. We will provide here a brief introduction to this field and we will present structures that may be found in MEMS design.

A membrane is a limiting case of a plate which is so thin that its stiffness is very slight, and in which the elastic properties of the system are determined by external forces, causing tension. Two-dimensional structures involved in MEMS devices have the features of plates. Often, less exactly, they are called membranes.

Longitudinal Waves in a Plate

The equation of motion for the longitudinal displacement u_x in a plate has the same form as Equation (1) valid for bars. The only difference concerns a phase velocity which is, in the case of a plate, equal to:

$$c_P = \sqrt{\frac{E}{\rho(1-\sigma^2)}} = \sqrt{\frac{E_P}{\rho}} \tag{24}$$

The quantity E_P is called **plate modulus**.

If the wavelength is similar in magnitude to a plate thickness, the elastic behaviour is more complex. As the theory describing this case has been introduced for the first time by Lamb, the waves propagated through plates of a thickness comparable to a wavelength are called Lamb waves. There are two independent waves propagating in parallel to the plate faces. In the first group there are waves with displacements distributed symmetrically around the plane *z=0*. The second one involves waves with displacements distributed anti-symmetrically around the plane *z=0*. These two forms, symmetrical and anti-symmetrical Lamb waves, are schematically shown in Figure 2.7.

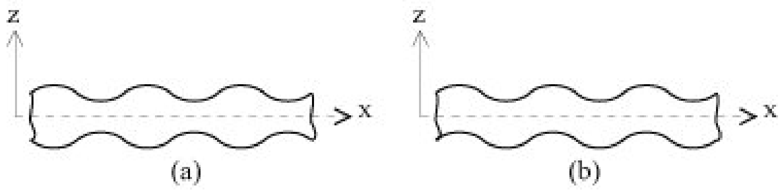


Figure 2.7 Symmetric (a) and anti-symmetric (b) Lamb wave.

There is a finite number of Lamb wave modes that can be generated in a plate of a given thickness and for a given frequency. This number depending on a plate thickness and on a frequency is obtained as a number of real roots of the characteristic equation that is obtained from the wave equation solution. Each of these waves has its phase velocity, group velocity, and stress and strain distribution profile.

The phase velocity is one of the fundamental characteristics of Lamb waves. Based on it, stresses and strains can be calculated at each point of the plate. Phase velocities for symmetrical (S_i) and anti-symmetrical (A_i) Lamb wave for the silicon (100) wafer are shown in Figure 2.8.

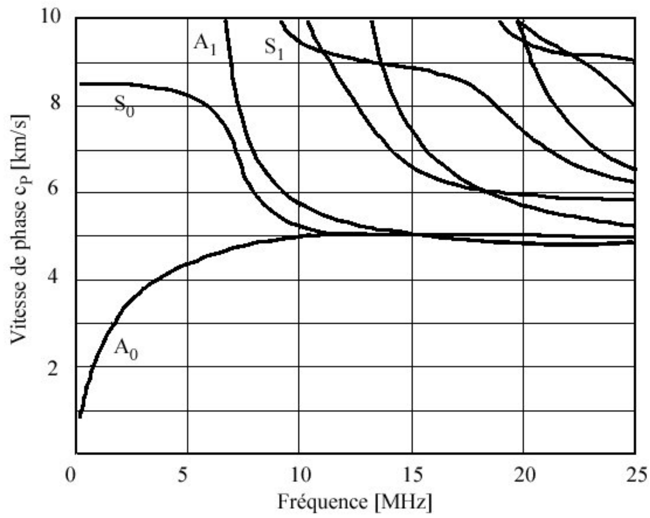


Figure 2.8 Phase velocity for symmetric (S_i) and anti-symmetric (A_i) Lamb wave.

For frequencies where the wavelength is great comparing to plate thickness, the phase velocity is equal to the Expression (24). For wavelengths much smaller than plate thickness, Rayleigh waves can be observed.

Transverse Waves in a Plate

There are many MEMS applications based on a transversally vibrating plate. We will assume, similarly as in the case of a beam, that the wavelength λ is larger than the plate thickness h (λ >> h). Under this assumption we can consider that the transverse cross-sections of the plate still remain planes in the presence of strains, and that no strains occur in the middle (neutral) line. The wave equation of the vibrations of the plate can be written in the form:

$$\left(B + \eta \frac{\partial}{\partial t}\right)\nabla^4 u = R'_B \frac{\partial u}{\partial t} + M \frac{\partial^2 u}{\partial t^2} \tag{25}$$

In this equation, B is the stiffness of the plate for bending:

$$B = \frac{Eh^3}{12\,(1-\sigma^2)} \tag{26}$$

M is the mass of the plate per unit surface, *h* are the specific losses of the plate, R'_B is the mechanical resistance of radiation losses referred to the unit area. In the first approximation, the losses are almost proportional to ?u/?t. By assuming sinusoidal vibrations and a plate without losses, Equation (23) can be written as the product:

$$\left(\nabla^2 u + k_B^2 u\right)\left(\nabla^2 u - k_B^2 u\right) = 0 \tag{27}$$

where $k_B = \omega/c_B$. The Equation (24) represents two component waves, the first of which is analogous to a propagating in a membrane. For a plate without losses ($R_B = 0$), the phase velocity of this component is:

$$c_B = \frac{\omega}{k_B} = \omega^{1/2}\left(\frac{B}{M}\right)^{1/4} \tag{28}$$

The second component is not related to the progressive wave, but represents a sinusoidal changing of local bending of the plate, which diminishes exponentially with an increased distance from the source. Due to this local reaction of the plate, it is not perfectly compliant for the point-acting force and the additional stiffness occurs on bending. This mechanical stiffness that does not appear in the case of a membrane (where B = 0), has a value:

$$s_m = \frac{8\omega}{(MB)^{-1/2}} \tag{29}$$

As in the case of bars, plates can be fixed on their edges in various ways. Generally, we can consider a set of conditions shown in Table 2.1. In MEMS devices, depending on the surrounding structure, a plate can be considered as clamped, and often, if its thickness, *h*, is much smaller than its lateral dimensions, the plate can be approximated as being hinged. Vibrations of a rectangular plate are usually discussed in a system of rectangular coordinates, *x*, *y*. The distribution of vibrations can be presented in the form of the product:

$$u = u_x(x)u_y(y) \tag{30}$$

If the plate of dimensions *a* and *b* is simply supported (hinged) on its edges, displacement components in x and y coordinate for any mode of vibration can be calculated from the following expressions:

$$\begin{aligned} u_x &= u_{x\,max} \sin\frac{m\pi x}{a}\exp(j\omega t), \quad m = 1,2,3,... \\ u_y &= u_{y\,max} \sin\frac{n\pi y}{b}\exp(j\omega t), \quad n = 1,2,3,... \end{aligned} \tag{31}$$

The frequencies of its free vibrations are given by:

$$f_{mn} = \frac{\pi}{2}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\sqrt{\frac{B}{M}} \tag{32}$$

The calculation of vibrations of a clamped plate is more complicated. We will give here the frequencies of vibrations corresponding to this kind of fixation:

$$f_{mn} = \frac{k_{xm}^2 + k_{yn}^2}{2\pi}\sqrt{\frac{B}{M}} \tag{33}$$

where

$$k_{xm}^2 \approx (2m+1)\pi/2a \quad , \qquad k_{yn}^2 \approx (2n+1)\pi/2b \tag{34}$$

The forms of first modes of vibration corresponding to the described plate fixations are schematically shown in Figure 2.9. Figure 2.10. shows, for a better understanding, a shape of two of these modes.

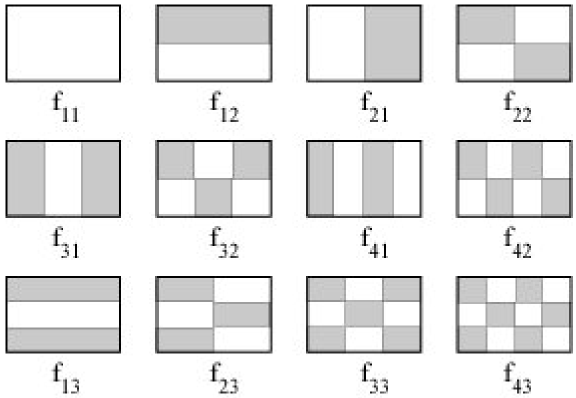
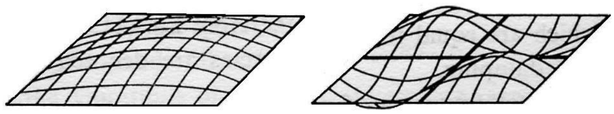


Figure 2.9 First vibration modes of a clamped rectangular plate.



(a) (b)

Figure 2.10 Shape of modes 11 (a) and 22 (b) of a clamped rectangular plate.

Although straight lines and rectangular shapes are typical for microelectronic design, circular forms are sometimes preferred because of their easier solution and symmetrical behaviour. In some cases, if the solution in the rectangular coordinates is lacking, we can thus apply a solution for a circular form to a square one as a first approximation. Harmonic vibrations of a plate are determined by the product of the circular and the radial distribution.

$$u = u_r(r)u_\varphi(\varphi) \tag{35}$$

The general formula for frequencies of free vibrations of plate with the radius *a* takes the form:

$$f_{mn} = \frac{\pi}{2a^2}\sqrt{\frac{B}{M}}\varepsilon_{mn} \tag{36}$$

The values of the coefficient e_{mn} for first modes of vibrations of a clamped circular plate are given in Table 2.3.

	m = 0	**m = 1**	**m = 2**
n = 1	1.015	1.468	1.879
n = 2	2.007	2.483	2.992
n = 3	3.000	3.490	4.000

Table 2.3 Coefficients ε_{mn} determining modes of vibrations of a clamped circular plate.

For a supported plate, the coefficient e_{01} corresponding to the mode *01* is equal to e_{01} = 0.507. We can deduce, from the comparison of the corresponding coefficients related to a clamped and supported circular plate, the ratio of the corresponding frequencies which is approximately equal to 2.

The knowledge of the distribution of stresses generated in the plate during bending is important in some applications. This helps in placing of piezoresistive gauges in a pressure sensor design for example. There are two stress components in plates. If we consider a clamped circular plate of a radius *a* and thickness *h* under the pressure *p*, the radial and tangential stress is:

$$T_r = \frac{3pza^2}{4h^3}\left[(1+\sigma)-(3+\sigma)\frac{r^2}{a^2}\right] \tag{37}$$

$$T_t = \frac{3pza^2}{4h^3}\left[(1+\sigma)-(1+3\sigma)\frac{r^2}{a^2}\right] \tag{38}$$

In these two expressions, r and z are the radial and vertical coordinate, respectively. The value of z = 0 corresponds to a central plane of the plate.

It should be emphasized that there is an effect of co-vibrating air on a plate mass plate and resistance. Under the assumption that the whole surface of the plate vibrates in phase like a piston, the values of additional mass and mechanical resistance can be calculated from a real and imaginary part of the radiating impedance. Values of a real part R_d, and imaginary part X_d of a radiation impedance of a circular plate and corresponding mass, *m* are the followings:

$$\begin{aligned} &R_d \sim \frac{8\rho\omega^4 a^6}{27\pi c^3}, \quad X_d \sim \frac{8\omega\rho a^3}{3}, \quad m \sim \frac{8\rho a^3}{3}, \quad \text{for ka} \ll 1 \\ &R_d \sim \pi a^2 \rho c, \quad X_d \sim 0, \quad m \sim 0, \quad \text{for ka} \gg 1 \end{aligned} \tag{39}$$

2.3 Three-Dimensional Systems

It is necessary, for a complete study of the propagation of mechanical waves in an anisotropic solid body, to describe its elastic properties in three dimensions. Such a description is done by means of a stress-strain relation. Since both stress and strain are second rank tensors, the most general linear relationship between stress and strain is a fourth rank tensor. We will define stress, strain and elastic coefficients that relate them. Later, we will describe briefly different types of waves that can propagate in solids.

The stress *T* is defined as the force *dF* per unit area *dA* acting on the surface of a differential volume element of a solid body. We can see in Figure 2.11 that there are three stress components on each surface of the element. Each of these components can be written in the following form:

$$T_{ik} = \frac{dF_i}{dA_k} \qquad i,k = x,y,z \tag{40}$$

where dF_i is a force component acting on the element surface dA_k. Stress components T_{xx}, T_{yy}, and T_{zz}, perpendicular to a differential plane, are called **normal stresses**. Stresses acting along the faces are called **shear stresses**. They are labelled with double subscripts, for example T_{xz}, the first subscript identifying the face and the second one identifying the direction.

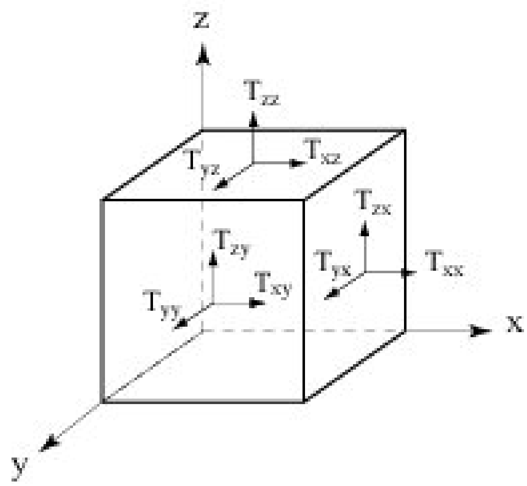


Figure 2.11 Stress components acting on a differential volume element.

Generally, stress is second rank tensor having nine components. In order to satisfy static equilibrium of the differential volume element, shear stresses are symmetric ($T_{ik} = T_{ki}$). This fact reduces the number of components of stress tensor to six and the matrix notation according to the following expression can be used.

$$[T] = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} T_{xx} \\ T_{yy} \\ T_{zz} \\ T_{yz} = T_{zy} \\ T_{xz} = T_{zx} \\ T_{xy} = T_{yx} \end{bmatrix} \tag{41}$$

When forces are applied to a solid body, it can be deformed. The differential deformation is called **strain**, expressed as change in length per unit length. The strain tensor in the matrix notation is shown in the following expression.

$$[S] = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ 2S_{yz} \\ 2S_{xz} \\ 2S_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \\ \partial/\partial y & \partial/\partial x & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{42}$$

In an ideal elastic body that is not piezoelectric, there exists a relation between stresses and strains that is given by a fourth rank tensor that in its most general form has 81 components. However, because of various symmetry arguments, there are many constraints among the 81 components, and, in every real material, there is a maximum of only 21 parameters describing its elasticity. These parameters can be written as elements of a square 6x6 symmetrical matrix. The stress-strain relation can be written in a form of the generalized Hooke's law as

$$[T] = [c][S] \quad , \qquad [S] = [s][T] \tag{43}$$

where *[c]* is a **stiffness matrix** and *[s]* is a **compliance matrix**. The number of 21 independent nonzero coefficients is reduced in many materials of practical interest. For example, in cubic materials such as a single-crystal silicon, there are only three independent quantities, and a great deal of symmetry. The stiffness coefficients of silicon are

$$[c] = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix} \tag{44}$$

The compliance matrix has the same form as the stiffness matrix; the nonzero stiffness coefficients are just replaced by the corresponding compliance coefficients. In the case of an isotropic material, elastic matrices have the highest degree of symmetry and have only two independent quantities. The Lamé's coefficients, *l* and *μ*, are often used to describe the elastic properties of isotropic materials. The stiffness coefficients of an isotropic material has the form

$$[c] = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{45}$$

In technical practise, Young's modulus and Poisson's ratio are more often used to describe material elastic properties. The relations between two couples of independent elastic constants are as follows

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \quad , \quad \sigma = \frac{\lambda}{2(\lambda+\mu)} \tag{46}$$

In Table 2.4, material constants that are currently used in microsystems are listed.
The dependence on the doping concentration, on the fabrication process, and on the isotropic degree explains the value ranges of elastic constants shown in Table 2.4.

Material	**Stiffness Coefficients [GPa]**			**Young's Modulus E [GPa]**	**Shear Modulus G [GPa]**	**Poisson's Ratio s [-]**	**Density r [10^3 kgm^{-3}]**
	c_{11}	**c_{12}**	**c_{44}**				
Silicon (Si)	165.6	63.9	79.5	132-190	54-91	0.04-0.23	2.329
Polysilicon	-	-	-	163-175	66-72	0.21-0.23	2.3
Oxide (SiO_2)	-	-	-	57-92	22-35	0.3	2.5
Nitride (Si_3N_4)	-	-	-	270-310	106-125	0.24-0.27	3.1
Gold (Au)	186	157	42	78.5	42	0.42	19.3
Aluminium (Al)	106.9	62.6	28.5	70	28.5	0.34	2.7

Table 2.4 Elastic constants of some materials used in MEMS.

Volume Waves in Solids

Longitudinal and transverse wave are the only two types of volume waves that can be generated in an infinite isotropic solid body. The longitudinal waves, called also P (primary) waves have the particle displacement in parallel to the wave propagation. The phase velocity of the longitudinal wave can be written as a function of the Young's modulus, Poisson's ratio, and density as:

$$c_L = \sqrt{\frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)}} \tag{47}$$

In transverse waves, called also shear waves, or S (secondary) waves, the particle displacements are perpendicular to the wave propagation. The phase velocity of the transverse wave can be written, based on the longitudinal velocity and Poisson's ratio, as:

$$c_T = c_L\sqrt{\frac{(1-2\sigma)}{(2-2\sigma)}} \tag{48}$$

Surface Waves

Rayleigh derived theoretically that a wave can propagate along the surface of a semi-infinite solid with a single boundary. In this surface wave, also called Rayleigh wave, the particles exercise an elliptical movement. The particle displacement can be decomposed in two components, longitudinal, u_x in parallel with the surface, and transversal, u_z perpendicular to the surface. There are other types of surface waves specific to different boundary conditions. There is, for instance, Bleustein-Gulyaev wave that is a transversal wave propagating on the surface of a piezoelectric semi-infinite solid. Lamb waves may be considered as a particular case of a surface wave propagating in a solid material limited by two planes whose distance is smaller than the wavelength. Stoneley waves can be generated at the surface of two semi-infinite isotropic solids. Love waves are transversal waves that can propagate in a thin layer deposited on a substrate.

Rayleigh wave velocity in a homogeneous substrate can be considered as frequency independent, for any practical situation, and can be approximated by the following expression:

$$c_R = c_T\frac{0.87 + 1.12\sigma}{1 + \sigma} \tag{49}$$

Expression (49) shows that the Rayleigh speed c_R is higher than the transversal speed c_T. Assuming that the value of the Poisson's ratio can theoretically be in the interval from 0 to 0.5, the Rayleigh speed can vary from 0.87 c_T to 0.96 c_T.
In an anisotropic solid, the velocity of the surface wave depends on the direction of propagation. The pure Rayleigh wave mode can appear only on specific crystal planes and in specific directions. Rayleigh waves propagate with a relatively small attenuation. The amplitudes of both components, longitudinal and transversal, decrease exponentially with the distance from the surface. The variation of relative amplitudes U_x and U_z as a function of the normalized distance z/λ_R in the case of a silicon crystal in the direction of propagation [100] on the plane (001) is shown in Figure 2.12.

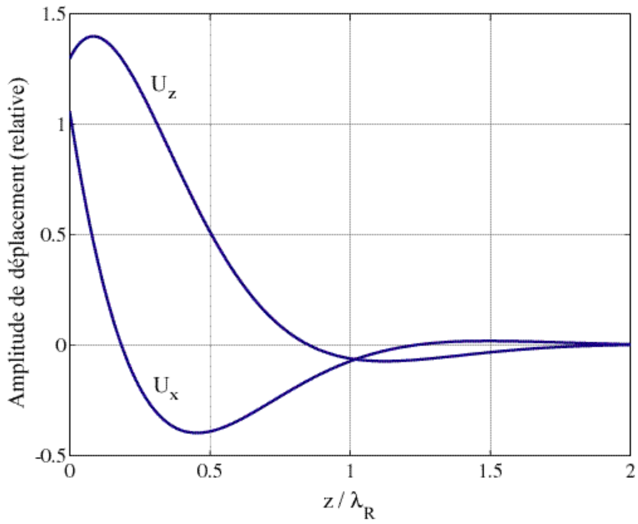


Figure 2.12 Components of particle displacement in surface wave in silicon.

2.4 Related Reading

[ARM 69] ARMANAKAS A. E., GAZIS D. C., HERRMANN G., *Free vibrations of circular cylindrical shells*, Pergamon Press, 1969.

[BRU 98] BRUNEAU M., *Manuel d'acoustique fondamentale*, Paris, Hermès, 1998.

[DIE 74] DIEULESAINT E., ROYER E., *Ondes élastiques dans les solides*, Paris, Masson et Cie, 1974.

[GRA 75] GRAFF K, *Wave Motion in Elastic Solids*, Dover, New York, 1975.

[HAN 99] HAN S. M., BENAROYA H., WEI T., «Dynamics of transversely vibrating beams using four engineering theories», J. Sound & Vibration, vol. 225, No.5, p. 935-988, 1999.

[MAL 69] MALECKI I., *Physical foundations of technical acoustics*, Pergamon, 1969.

[MER 81] MERHAUT J., *Theory of electroacoustics*, McGraw-Hill, 1981.

[MID 89] MIDDELHOEK, S., AUDET, S. A., *Silicon sensors*, Academic Press, 1989.

[MOR 50] MORSE R. W., «Velocity of compressional waves in rods of rectangular cross-section», J. Acoust. Soc. America 22, 219.

[PET 82] PETERSEN K. E., «Silicon as a mechanical material », *Proceedings of the IEEE*, vol. 70, n°. 5, 1982.

[POL 77] POLLARD H. F., *Sound waves in solids*, London, Pion Ltd., 1977.

[RAI 97] RAI-CHOUDHURY P., *Handbook of Microlithography, Micromachining, and Microfabrication*, vol. 2, London, SPIE, 1997.

[SEN 01] SENTURIA S. D., *Microsystem Design*, Kluwer Academic Publishers, 2001.

[SKV 91] SKVOR Z., *Vibrating systems and their equivalent circuits*, Prague, Academia, 1991.

[TIL 96] TILMANS H.A.C., «Equivalent circuit representation of electromechanical transducers: I. Lumped-parameter systems», *J. Micromech. Microeng.*, vol. 6, p. 157-176, 1996.

[TIL 97] TILMANS H.A.C., «Equivalent circuit representation of electromechanical transducers: II. Distributed-parameter systems», *J. Micromech. Microeng.*, vol. 7, p. 285-309, 1997.

[TIM 61] TIMOSHENKO S., WOINOWSKY-KRIEGER S., *Théorie des plaques et coques*, Librairie Polytechnique Ch. Béranger, 1961.

[VIK 67] VIKTOROV I. A., *Rayleigh and Lamb waves*, N. York, Plenum Press, 1967.

[WHI 70] WHITE R. M., «Surface elastic waves», *Proc. IEEE*, vol. 58, p. 1238-1276, 1970.

[YAR 00] YARALIOGLU, G.G., DEGERTEKIN, F.L., BADI, M.H., AULD, B.A., KHURI-YAKUB,B.T., «Finite element method and normal mode modeling of capacitive micromachined SAW and Lamb wave transducers», *Ultrasonics Symp.*, San Juan, Puerto Rico, 2000.