1. Lumped-elements Systems

Modelling and/or analysis methods based on lumped-elements can be applied on systems that can be considered as linear and without any propagation effects. The first condition can be fulfilled under small-signal conditions, the second one introduces the limits for system dimensions to wavelength ratio that must be sufficiently smaller than 1. In this case, a mechanical system can be decomposed into a certain number of ideal elements carrying uniquely the elastic or potential energy or being dissipative. In this way, the system can be replaced by non-deformable and non-dissipative mass elements, by zero-mass and non-dissipative elastic elements, and by non-deformable and zero-mass dissipative elements. These elements are analogous to electrical inductors, capacitors, and resistors; therefore all mathematical tools developed for these electrical elements can be adopted for lumped-elements mechanical systems. Various circuit elements connected at nodes form a circuit that is governed by generalizations of **Kirchhoff's Laws**. We will use the concept of conjugate power variables to quantify the relations within a circuit. In the most general case, the two conjugate power variables are an ***effort*** *e(t)* and a ***flow*** *f(t)*. The dimension of the product *e.f* is a power. These domain-specific variables are linked with variables describing the element behaviour. There are five types of ideal one-port elements and there are also elements with multiple terminals. Two variables assigned to element terminals are ***through* variable** and ***across* variable**. A *through* variable traversing an element can be associated with the current (or *flow*) in electric circuits and an *across* variable identified between two terminals corresponds to the voltage (or *effort*). There is a number of conventions used for assignments of variables. The preceding example from electrical circuits corresponds to the 'effort-to-voltage' convention. The convention 'flow-to-voltage' is used in mechanical circuits, while the 'effort-to-voltage' convention is used in acoustic circuits. There is also the convention called 'HDL' that is used when formulating abstract circuit elements in hardware-description languages. In this case, an *effort* and a displacement are used as a pair of system variables. A displacement is defined, based on a *flow* variable, as:

$$q(t) = \int_{t_0}^{t} f(t)dt + q(t_0) \qquad (1)$$

In the 'HDL' convention, a displacement is assigned as the *across* variable, an *effort* as the *through* variable and the product *e.f* corresponds to energy. In the thermal energy domain, the temperature, *T*, is assigned as the across variable and the heat current, *dQ/dt*, is the through variable. Table 1.1 summarizes different conventions discussed above and shows their basic features.

Convention	Across Variable	Through Variable	Product	Principal Use
Effort-to-Voltage	e	f	Power	Electrical Circuits Acoustic Circuits Mechanical Circuits
Flow-to-Voltage	f	e	Power	Mechanical Circuits
HDL	q	e	Energy	Mechanical Circuits in HDL Representation
Thermal	T	dQ/dt	Watt-Kelvin	Thermal Circuits

Table 1.1 Conventions used in different domains.

The choice of convention used for a given physical domain and for a given type of device is not unique. Any of the conventions can be used to reach the same conclusions about the device behaviour. It is therefore important to check the assignment convention used by other authors when comparing their approaches.

Kirchhoff's Laws, in its general form, reveal a unique relationship between *efforts*, *flows*, and element characteristics. Kirchhoff's Voltage Law (KVL) states that the oriented sum of all *across* variables, a_i, around any closed loop is zero. KVL can be expressed by the following expression:

$$\sum a_i = 0 \tag{2}$$

Kirchhoff's Current Law (KCL), saying that the sum of all *through* variables, b_i, entering a node is zero, can be expressed as:

$$\sum b_i = 0 \tag{3}$$

Lumped elements can be classified according to the number of terminals used to join them to a circuit. A pair of terminals that carry the same *through* variable is called a port. There are three basic one-port elements, and two independent source elements, that form any circuit. For derivator or generalized inductor, *L*, the *across* variable is proportional to the derivative of the *through* variable:

$$a = L\frac{d(b)}{dt} \tag{4}$$

The integrator or generalized capacitor, *C*, is a one-port element for which the *across* variable is proportional to the integral of the *through* variable:

$$a = \frac{1}{C}\int b\,dt \tag{5}$$

The generalized resistor, *R*, is defined by the relation between the *across* variable and the *through* variable:

$$a = R\,b \tag{6}$$

The symbols used for the three basic one-port elements are shown in Figure 1.1.

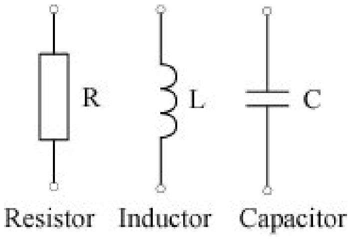

Figure 1.1 Basic one-port elements in the 'effort-to-voltage' convention.

The ratio of the Laplace transform of the *across* variable to the Laplace transform of the *through* variable is called immitance. This relation can be written, for the case of harmonic variables, as a ratio of respective phasors:

$$V = \frac{\hat{a}}{\hat{b}} \tag{7}$$

In the 'effort-to-voltage' convention, the immitance is equal to the impedance; in the 'flow-to-voltage' convention, the immitance is equal to the admittance.

Figure 1.2 shows two independent source elements, source of the across variable and source of the through variable.

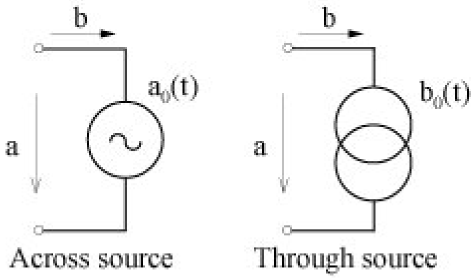

Figure 1.2 Independent source elements in the circuit representation.

Among the two-port elements there are in particular a **transformer** and a **gyrator** that are useful especially in the theory of transducers for translating the variables from one energy domain to another. They both are lossless and memoryless. The total power entering the element through both ports of a lossless two-port must be equal to zero. The element must therefore satisfy the following equation:

$$a_1 b_1 + a_2 b_2 = 0 \tag{8}$$

Both, the definition of a transformer, shown here in the form of the cascade matrix:

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1/N_t & 0 \\ 0 & -N_t \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \tag{9}$$

and that of a gyrator:

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 & -N_g \\ 1/N_g & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \tag{10}$$

satisfy the condition for a lossless two-port element. Circuit symbols used for these two-port elements are shown in Figure 1.3.

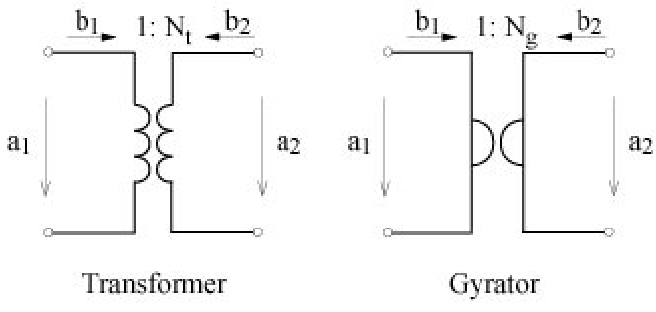

Figure 1.3 Transformer and gyrator circuit symbols.

1.1 Electro-mechanical Analogy

The electro-mechanical analogy is developed under the hypothesis that the system is composed of ideal components carrying either the kinetic or potential energy or are of purely dissipative behaviour. Such components, separated in the space, are represented by masses localized at a point, perfectly elastic mass-less springs, and dissipative elements. This approximation can be used under the condition that these components are much smaller than the wavelength of an acting signal. A mechanical system can thus be symbolized by a circuit composed of three basic elements: mass, compliance and resistance (see Fig. 1.4).

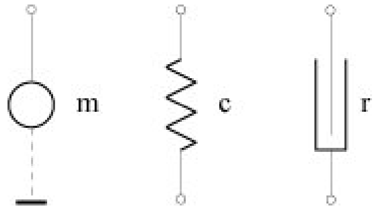

Figure 1.4 Symbolic elements of a mechanical system.

Force and velocity are two basic variables used in the description of translational mechanical systems. The relation between these two variables applied to the mass is given by the Newton's Law:

$$F = m\frac{dv}{dt} \tag{11}$$

The relation valid for an elastic element obeying the Hooke's Law is:

$$F = k\,u = \frac{1}{c}\int v dt \tag{12}$$

where *u* is the displacement, and *k* is the spring factor. We can notice the simple relation between the compliance and the spring factor:

$$c = 1/k \tag{13}$$

For friction losses we can write the relation between force and velocity in the form:

$$F = rv \tag{14}$$

We can notice the similarity between the expressions (11), (12), (14), and (4), (5), (6). If considering the force as an *effort* variable and the velocity as a *flow* variable, the Effort-to-Voltage convention can be used for the description of a mechanical system by an analogue circuit. This convention, also called direct electro-mechanical analogy is summarized in the following table:

Mechanical Variable		**Electrical Variable**	
Force	F[N]	Voltage	u [V]
Velocity	v [m/s]	Courant	i [A]
Displacement	u [m]	Charge	q [C]
Mass	m [kg]	Inductance	L [H]
Compliance	c [m/N]	Capacity	C [F]
Resistance	r [kg/s]	Resistance	R [Ω]

Table 1.2 Relations of direct electro-mechanical analogy.

From the direct electro-mechanical analogy, we can deduce an equivalent circuit which is analogue to the symbolic circuit describing the mechanical system. The duality between symbolic and analogue circuit must be taken into account when creating the analogue circuit for the system analysis. This results in replacing the elements that share a common flow by elements connected in series in an equivalent circuit; and in connecting in parallel the elements that share a common effort. Figure 1.5 (a) shows spring-mass-dashpot assembly where all three elements share the same flow and thus a common variation of displacement. The correct equivalent circuit is shown in Figure 1.5 (b).

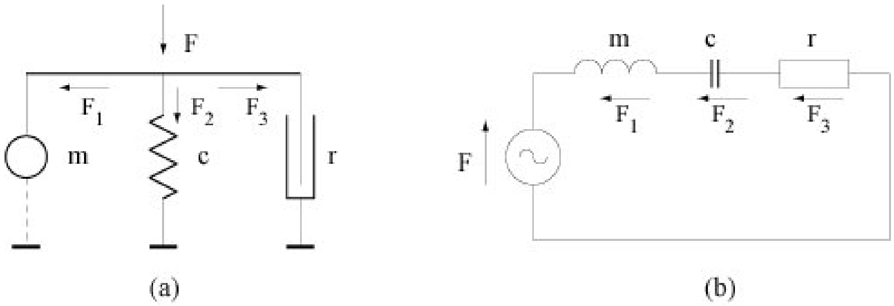

Figure 1.5 Symbolic (a) and equivalent (b) circuits of a mechanical system with shared flow.

The immitance defined by the Expression (6) is in the case of the direct electro-mechanical analogy equal to the mechanical impedance. Its unit is mechanical Ohm and its dimension is Nsm^{-1}. The mechanical admittance, the inverse value of mechanical impedance, is often called *mobility*.

1.2 Electro-acoustic Analogy

Acoustic lumped elements can be defined based on the hypothesis of the sound wavelength being long compared to the relevant physical length scales of the problem. Similarly as in the case of a mechanical system, three basic elements of a lumped acoustical system are: acoustic mass, m_a, acoustic compliance, c_a, and acoustic resistance r_a. Figure 1.6 shows a representation of these elements in a symbolic diagram.

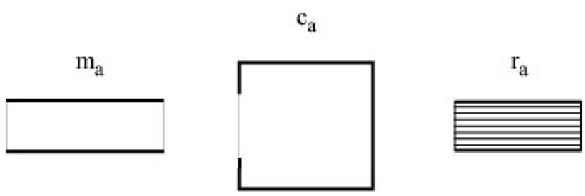

Figure 1.6 Symbolic elements of an acoustic system.

Acoustic pressure, *p*, and volume velocity, $w = vA$, are two basic variables used in the description of acoustic lumped element systems. The acoustical impedance, Z_a, is defined, similarly as in the preceding case, as:

$$Z_a = \frac{p}{w} \tag{15}$$

Let us examine the effect of a simple harmonic acoustic wave, interacting with a tube of length *l* and cross-sectional area *A*. Pressure disturbance, *p*, across the ends of a tube caused by this acoustic wave will accelerate the mass of gas according to Newton’s law:

$$p = \frac{m}{A^2}\frac{dw}{dt} = m_a\frac{dw}{dt} \tag{16}$$

Acoustic mass, m_a, can thus be expressed as:

$$m_a = \frac{\rho_0 l}{A} \tag{17}$$

Acoustic compliance is related to the elastic property of a gas enclosed in a volume V_0. By using the relationship between volume and pressure for an adiabatic change in a gas, we can write an expression relating the pressure change, *p*, inside the volume V_0 to the displacement, *d*, at its input:

$$p = \frac{\gamma P_0 A d}{V_0} \tag{18}$$

where g is the ratio of the specific heats at constant pressure and volume. We assume that the volume change produced by the displacement *d* is small compared to the volume V_0. The relation between displacement and velocity for a simple harmonic motion is:

$$d = \frac{v}{j\omega} \tag{19}$$

The acoustic impedance of an enclosed volume is then:

$$Z_a = \frac{\gamma P_0}{j\omega V_0} = \frac{1}{j\omega c_a} \tag{20}$$

We can see from the preceding expression that the acoustic compliance is:

$$c_a = \frac{V_0}{\gamma P_0} \tag{21}$$

Acoustic resistance is an ideal component that represents either the transformation of the acoustic energy to the other form of energy or a radiation of the acoustic energy from the system. An acoustic resistance can be realized either by a tube or a slit (see Figure 1.7) where energy is lost because of viscous dissipation.

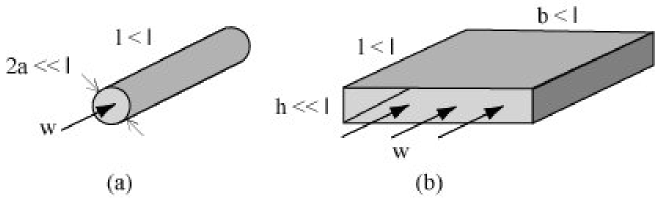

Figure 1.7 Capillary tube (a) and capillary slit (b).

The relation between acoustic pressure, volume velocity and acoustic resistance, r_a, is in the case of viscous losses given by:

$$p = r_a w \tag{22}$$

When laminar flow conditions exist and the acoustic wavelength is long compared to the tube or slit length, then the acoustic resistance of slit with length *l*, width *b*, and height *h* is:

$$r_a = \frac{12\mu l}{h^3 b} \tag{23}$$

where μ is viscosity of gas. In the case of a tube of radius a, and length l, the acoustic resistance is:

$$r_a = \frac{8\mu l}{\pi a^4} \tag{24}$$

The acoustic impedance of a slit or of a tube can be generally considered as a complex value composed of a real part corresponding to a resistance and of an imaginary part given by a reactive component, mass. For the harmonic motion we can write:

$$Z_a = r_a + j\omega m_a \tag{25}$$

For capillary tubes and slits, the acoustic mass is greater than that derived without considering a viscosity and given by the Expression (17). The corrected value of a mass of a slit is then:

$$m_a = 1.2\frac{\rho_0 l}{hb} \tag{26}$$

In the case of a capillary tube, the mass is:

$$m_a = \frac{4}{3}\frac{\rho_0 l}{\pi a^2} \tag{27}$$

There are limiting values for critical dimensions of slits and tubes that ensure their resistive behaviour. In this case, the Expression (25) will be predominantly real. Supposing the propagating medium in the acoustic structure is the air, the limiting conditions for a slit height *h* and for a tube radius *a* at the maximal frequency are:

$$h \ll \frac{5\,10^{-3}}{\sqrt{f}}, \quad a \ll \frac{3\,10^{-3}}{\sqrt{f}} \tag{28}$$

We will consider the pressure as an *effort* variable and the volume velocity as a *flow* variable. The 'effort-to-voltage' convention will be used for the description of an acoustic system by an analogue circuit. This convention, also called electro-acoustic analogy, is summarized in the following table:

Acoustic Variable		**Electrical Variable**	
Acoustic Pressure	P [Pa]	Voltage	u [V]
Volume Velocity	w [m^3/s]	Courant	i [A]
Volume Displacement	X [m^3]	Charge	q [C]
Acoustic Mass	m [kg/m^4]	Inductance	L [H]
Acoustic Compliance	C [m^5/N]	Capacity	C [F]
Acoustic Resistance	R [kg/m^4s]	Resistance	R [?]

Table 1.3 Relations of the electro-acoustic analogy.

Building an equivalent model, based on a symbolic diagram, is easier in the case of an acoustic system than in a mechanical one. All holes and slits can be directly replaced by masses and resistances and all closed volumes are replaced by compliances connected in one end with a common reference point. The transition from a symbolic to an equivalent circuit can be deduced from the example shown in Figure 1.8.

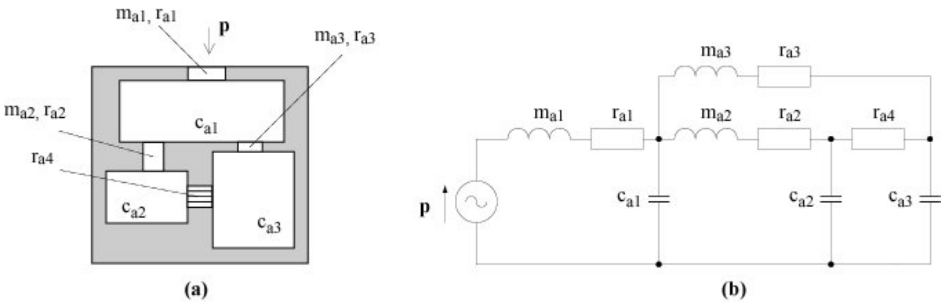

Figure 1.8 Symbolic circuit (a) and equivalent circuit (b) of an acoustic system.

1.3 Related Reading

[BRU 98] BRUNEAU M., *Manuel d'acoustique fondamentale*, Paris, Hermès, 1998.

[JOU 94] JOUHANEAU J., *Notions élémentaires d'acoustique – Electroacoustique*, Paris, Tec&Doc, 1994.

[MER 81] MERHAUT J., *Theory of electroacoustics*, McGraw-Hill, 1981.

[OLS 57] OLSON H. F., *Acoustical engineering*, D. van Nostrand Comp.,Inc., 1957.

[ROS 86] ROSSI M., *Electroacoustique*, Presses Polytechniques Romandes, 1986.

[RUF 02] RUFER L., *La modélisation des microsystèmes électromécaniques*. in Mir, S. (Ed.), *Conception des microsystèmes sur silicium*. Traité EGEM, Hermes Science Publications, pp. 101-128, 2002.

[SEN 01] SENTURIA S. D., *Microsystem Design*, Kluwer Academic Publishers, 2001.

[TIL 96] TILMANS H.A.C., «Equivalent circuit representation of electromechanical transducers: I. Lumped-parameter systems», *J. Micromech. Microeng.*, vol. 6, pp. 157-176, 1996.

[TIL 97] TILMANS H.A.C., «Equivalent circuit representation of electromechanical transducers: II. Distributed-parameter systems», *J. Micromech. Microeng.*, vol. 7, pp. 285-309, 1997.